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An integrable model in general relativity

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Abstract. It is shown that gravitational fields invariant for a non Abelian 2-dimensional Lie algebra of Killing fields are parameterized either by solutions of a transcendental equation (the *tortoise equation*) or by solutions of a linear second order partial differential equation (the *Laplace equation* or the *Darboux equation)* on the plane. Those determined *via* Laplace or Darboux equations are exact nonlinear gravitational waves obeying to two nonlinear superposition laws.

PACS. 04.20.-q Classical general relativity – 04.20.Gz Spacetime topology, causal structure, spinor structure – 04.20.Jb Exact solutions

1 Introduction

In the last years, together with the experimental efforts devoted to the detection of gravitational waves, there is a strongly related theoretical activity to describe and predict the emission of gravitational waves from astrophysical systems in strong field conditions. However, all the experimental devices, laser interferometers (*e.g.* GEO-600, VIRGO, LIGO), or resonant antennas, are constructed coherently with results obtained from the non covariant linearized Einstein field equations, in close analogy with that is normally done in Maxwell theory of electromagnetic fields. Thus, a great deal of interest is still devoted to explicit solutions which more easily enable to discriminate between a physical or pathological feature.

Moreover, starting from the seventy's, new powerful mathematical methods have been invented to deal with nonlinear evolution equations and their exact solutions (see [9] and references therein).

In 1978, Belinskii and Zakharov considered a metric of the form

$$g = f(z,t)\left(\mathrm{d}t^2 - \mathrm{d}z^2\right) + h_{11}(z,t)\,\mathrm{d}x^2 + h_{22}(z,t)\,\mathrm{d}y^2 + 2h_{12}(z,t)\,\mathrm{d}x\mathrm{d}y.$$

The corresponding Einstein equations reduce essentially[1] to

$$\left(\alpha \mathbf{H}^{-1}\mathbf{H}_{\xi}\right)_{\eta} + \left(\alpha \mathbf{H}^{-1}\mathbf{H}_{\eta}\right)_{\xi} = 0,$$

where $\mathbf{H} \equiv \|h_{ab}\|$ and

$$\sqrt{2}\xi = (t+z)\,, \quad \sqrt{2}\eta = (t-z)\,, \quad \alpha = \sqrt{|\det \mathbf{H}|}.$$

A suitable generalization of the *Inverse Scattering Transform*, allows to solve the above equation and then to obtain *solitary waves solutions* [1], as for instance

$$\begin{aligned}-\mathrm{d}s^2 = {} & \frac{C_1^2 z^{2q^2}\cosh(qr+C_2)}{\left[t^2-z^2\right]^{1/2}}\left(\mathrm{d}t^2-\mathrm{d}z^2\right) \\ & + \frac{\cosh(s_1 r + C_2)}{\cosh(qr+C_2)} t^{2s_1}\mathrm{d}x^2 + \frac{\cosh(s_2 r - C_2)}{\cosh(qr+C_2)} z^{2s_2}\mathrm{d}y^2 \\ & - \frac{2\sinh(r/2)}{\cosh(qr+C_2)} z\,\mathrm{d}x\mathrm{d}y,\end{aligned}$$

with $t^2 \geq z^2$ and where s_1 and s_2 are constants satisfying the condition $s_1 + s_2 = 1$, so that they can be expressed, in terms of one arbitrary constant parameter q, as $s_1 = 1/2 + q, \quad s_2 = 1/2 - q$. The function r is defined by:

$$\exp r = 2z^{-2}t^2 - 1 - 2\left[z^{-2}t^2\left(z^{-2}t^2 - 1\right)\right]^{1/2}.$$

It can be easily verified that for any t the extremum of g_{11} with respect to the spacelike coordinate z, will correspond to the same constant value r_0 of the function r. Then, the world line of the extremum has the equation $t = z\cosh(r_0/2)$, and therefore the speed of this localized disturbance is smaller than the light velocity.

A geometric inspection of the above metric shows that it is invariant under translations along x, y-axis, *i.e.* it has the vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ as Killing fields which, since $\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] = 0$, close a Abelian 2-dimensional Lie algebra[2] $\mathcal{A}_2$. Moreover, the distribution $\mathcal{D}$, generated by $\frac{\partial}{\partial x}$

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[1] The function f can be obtained by quadratures in terms of the matrix $\mathbf{H}$.

[2] The study of Einstein metrics invariant for a Abelian 2-dimensional Killing Lie algebra goes back to Einstein and Rosen [3,6], Kompaneyets [5].

and $\frac{\partial}{\partial y}$, is 2-dimensional and the distribution $\mathcal{D}^{\perp}$ orthogonal to $\mathcal{D}$, is integrable[3] and transversal to $\mathcal{D}$.

Thus, it is natural to consider the general problem of characterizing all gravitational fields g admitting a Lie algebra $\mathcal{G}$ of Killing fields such that:

I. the distribution $\mathcal{D}$, generated by the vector fields of $\mathcal{G}$, is 2-dimensional;
II. the distribution $\mathcal{D}^{\perp}$ orthogonal to $\mathcal{D}$, is integrable and transversal to $\mathcal{D}$.

A 2-dimensional $\mathcal{G}$, is either Abelian ($\mathcal{A}_2$) or non-Abelian ($\mathcal{G}_2$). A metric g satisfying I and II, with $\mathcal{G} = \mathcal{A}_2$, or $\mathcal{G}_2$ will be called *$\mathcal{G}$-integrable*.

In the following, $\mathcal{K}il\,(g)$ will denote the Lie algebra of all Killing fields of a metric g while *Killing algebra* will denote a sub-algebra of $\mathcal{K}il\,(g)$.

2 Invariant metrics

2.1 Semi-adapted coordinates

Let g be a metric on the space-time M and $\mathcal{G}_2$ one of its Killing algebras whose generators X, Y satisfy

$$[X, Y] = sY, \quad s = 0, 1. \tag{1}$$

The Frobenius distribution $\mathcal{D}$ generated by $\mathcal{G}_2$ is 2 -dimensional and a chart (x^1, x^2, x^3, x^4) exists such that

$$X = \frac{\partial}{\partial x^3}, \quad Y = \left(\exp s x^3\right) \frac{\partial}{\partial x^4} \cdot \tag{2}$$

Such a chart will be called *semiadapted* (to Killing fields).

2.2 Invariant metrics

It can be easily verified [8] that in a semiadapted chart g has the form

$$\begin{aligned} g = {} & g_{ij} \mathrm{d}x^i \mathrm{d}x^j + 2\left(l_i + s m_i x^4\right) \mathrm{d}x^i \mathrm{d}x^3 - 2 m_i \mathrm{d}x^i \mathrm{d}x^4 \\ & + \left(s^2 \lambda \left(x^4\right)^2 - 2 s \mu x^4 + \nu\right) \mathrm{d}x^3 \mathrm{d}x^3 \\ & + 2\left(\mu - s\lambda x^4\right) \mathrm{d}x^3 \mathrm{d}x^4 + \lambda \mathrm{d}x^4 \mathrm{d}x^4, \quad i, j = 1, 2; \end{aligned}$$

with g_{ij}, m_i, l_i, λ, μ, ν arbitrary functions of $\left(x^1, x^2\right)$.

2.3 Killing leaves

Condition II allows to construct semi-adapted charts, with new coordinates $\left(x, y, x^3, x^4\right)$, such that the fields $e_1 = \frac{\partial}{\partial x}$, $e_2 = \frac{\partial}{\partial y}$, belong to $\mathcal{D}^{\perp}$. In such a chart, called from now on *adapted*, the components l_i's and m_i's vanish.

We will call *Killing leaf* an integral (bidimensional) submanifold of $\mathcal{D}$ and *orthogonal leaf* an integral (bidimensional) submanifold of $\mathcal{D}^{\perp}$. Since $\mathcal{D}^{\perp}$ is transversal to $\mathcal{D}$, the restriction of g to any Killing leaf, S, is non-degenerate. Thus, $(S, g|_S)$ is a homogeneous bidimensional Riemannian manifold. Then, the Gauss curvature $K\,(S)$ of the Killing leaves is constant (depending on the leave). In the chart $(p = x^3|_S,\ q = x^4|_S)$ one has

$$g|_S = \left(s^2 \widetilde{\lambda} q^2 - 2 s \widetilde{\mu} q + \widetilde{\nu}\right) \mathrm{d}p^2 + 2\left(\widetilde{\mu} - s\widetilde{\lambda} q\right) \mathrm{d}p \mathrm{d}q + \widetilde{\lambda} \mathrm{d}q^2,$$

where $\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu}$, being the restrictions to S of λ, μ, ν, are constants, and

$$K\,(S) = \widetilde{\lambda} s^2 \left(\widetilde{\mu}^2 - \widetilde{\lambda}\widetilde{\nu}\right)^{-1}. \tag{3}$$

This shows that the following cases can occur for $(S, g|_S)$.

1. $\widetilde{\lambda} > 0$, $\widetilde{\lambda}\widetilde{\nu} - \widetilde{\mu}^2 < 0$: $(S, g|_S)$ is a non-Euclidean plane, *i.e.* a bidimensional Riemannian manifold of negative constant Gauss curvature.
2. $\widetilde{\lambda} < 0$, $\widetilde{\lambda}\widetilde{\nu} - \widetilde{\mu}^2 > 0$: $(S, g|_S)$ is an "anti" non-Euclidean plane, *i.e.* is endowed with the metric of the previous case multiplied by -1.
3. $\widetilde{\lambda}\widetilde{\nu} - \widetilde{\mu}^2 < 0$: $(S, g|_S)$ is any indefinite bidimensional metric of constant Gauss curvature.

Since the Killing leaves are parametrized by x, y, the function

$$K = K\,(x, y) = \lambda s^2 \left(\mu^2 - \lambda\nu\right)^{-1}$$

describes the behavior of the Gauss curvature when passing from one Killing leave to another.

It is worth to note that the Killing algebra $\mathcal{G}_2$ is a subalgebra of the algebra $\mathcal{K}il\,(g_0)$, g_0 being a bidimensional metric of constant curvature (for instance, $g_0 = g|_S$).

If g_0 is positive (respectively, negative) definite and of positive (respectively, negative) Gauss curvature, then $\mathcal{K}il\,(g_0)$ is isomorphic to $so\,(3)$. But $so\,(3)$ does not admit bidimensional subalgebras at all. This explains why $g|_S$ cannot be a positively (respectively, negative) curved metric in the case (1) (respectively, (2)).

Similarly, if g_0 is a positive or negative definite flat metric, then $\mathcal{K}il\,(g_0)$ admits only Abelian bidimensional subalgebras. This explains why both positive and negative definite flat metrics are absent in the above list for $g|_S$.

In all other cases, the algebra $\mathcal{K}il\,(g_0)$ admits bidimensional non-Abelian subalgebras.

More exactly, if g_0 is not flat, then $\mathcal{K}il(g_0)$ is isomorphic to $so\,(2,1)$. Let $\mathbf{g}$ be the Killing form of $so\,(2,1)$. Then, the tangent planes to the isotropic cone of $\mathbf{g}$ exhaust the bidimensional non-Abelian Lie subalgebras of $so\,(2,1)$. If g_0 is flat and, thus, indefinite, then any bidimensional subspace of the algebra $\mathcal{K}il\,(g_0)$ different from its *commutator,* which is Abelian, is a non-Abelian subalgebra.

It is not difficult to describe the algebra $\mathcal{K}il\,(g|_S)$ in the semi-adapted coordinates (p, q). A direct computation shows that $\mathcal{K}il(g_0)$ has the following basis:

$$\widetilde{X} = \partial_p,\ \widetilde{Y} = \mathrm{e}^{sp}\partial_q,$$

[3] The integrability of the orthogonal distribution follows from the Abelian character of the Killing Lie algebra.

$$\widetilde{Z} = \mathrm{e}^{-sp}\left[2\left(s\widetilde{\lambda}q - \widetilde{\mu}\right)\partial_p + \left(s^2\widetilde{\lambda}q^2 - 2s\widetilde{\mu}q + \widetilde{\nu}\right)\partial_q\right],$$

$$\left[\widetilde{X}, \widetilde{Y}\right] = s\widetilde{Y},\ \left[\widetilde{X}, \widetilde{Z}\right] = -s\widetilde{Z},\ \left[\widetilde{Y}, \widetilde{Z}\right] = 2s\widetilde{\lambda}\widetilde{X}.$$

In the case $\lambda = 0$, the metric $g|_S$ is flat indefinite and it is convenient to identify $(S, g|_S)$ with the standard plane $\left(R^2, \mathrm{d}\xi^2 - \mathrm{d}\eta^2\right)$, $R^2 = \{(\xi,\eta)\}$. To do that it is necessary to choose a bidimensional non-commutative subalgebra in $\mathcal{K}il\left(\mathrm{d}\xi^2 - \mathrm{d}\eta^2\right)$ (they are all equivalent). For instance, by choosing $Y_0 = \partial_\xi + \partial_\eta$, $X_0 = -\eta\partial_\xi - \xi\partial_\eta$, we have $[X_0, Y_0] = Y_0$, $X_0, Y_0 \in \mathcal{K}il\left(\mathrm{d}\xi^2 - \mathrm{d}\eta^2\right)$ and, for $s \neq 0$, one can identify the quadruple $\left(S,\ 2\left(\mathrm{d}p\mathrm{d}q - q\mathrm{d}p^2\right),\ X|_S,\ Y|_S\right)$ with $\left(R^2,\ \mathrm{d}\xi^2 - \mathrm{d}\eta^2,\ X_0,\ Y_0\right)$.

The simply connected Lie group G corresponding to $\mathcal{G}$ is isomorphic to the group of affine transformations of R^2. Then, both S and R^2 are diffeomorphic to G as homogeneous $G-$spaces and the above identification of them is an equivalence of $G-$spaces.

The Killing form of $\mathcal{G}$ determines naturally a symmetric covariant tensor field on the $G-$space G which is identified with $\mathrm{d}\widetilde{x}^2$ on S and with $(\xi-\eta)^{-2}(\mathrm{d}\xi - \mathrm{d}\eta)^2$ on R^2. We will continue to call it *Killing form.* Thus, in the above identification the metric $g|_S$ for $\lambda = 0$ and $s = 0$ corresponds to

$$\widetilde{\mu}\left(\mathrm{d}\xi^2 - \mathrm{d}\eta^2\right) + \widetilde{\nu}\,(\xi-\eta)^{-2}(\mathrm{d}\xi - \mathrm{d}\eta)^2. \tag{4}$$

This representation of the metric $g|_S$ has been used to describe global solutions of the Einstein equations in [8].

3 The Ricci tensor field

In this section the notation $x^1 = x$, $x^2 = y$, $x^3 = p$, $x^4 = q$, will be used; moreover, Greek letters indices take values from 1 to 4; the first Latin letters indices take values from 3 to 4, while i, j from 1 to 2.

Let g be a $\mathcal{G}_2$-integrable metric, such that the matrix $\mathbf{M}_\Theta(g)$ associated to g is of the form

$$\mathbf{M}_\Theta(g) = diag\,(\mathbf{F}, \mathbf{H}) \tag{5}$$

where $\mathbf{F}$ and $\mathbf{H}$ are 2×2 matrices whose elements depend only on x^1 and x^2. It is suitable to distinguish two cases according to whether $\mathbf{F}$, *i.e.,* the matrix associated to the metric restricted to $\mathcal{D}^\perp$, has negative or positive determinant.

If $\det\mathbf{F} < 0$, then the components of the Ricci tensor in a non-holonomic adapted basis are

$$(R_{ab}) = \frac{\mathbf{H}}{2f\alpha}\left[(\alpha\mathbf{A}_{,1})_{,2} + (\alpha\mathbf{A}_{,2})_{,1}\right] + \frac{s^2}{\alpha^2}\mathbf{H}h_{22},$$
$$R_{12} = \partial_1\partial_2\ln\alpha\,|f| + \frac{1}{4}\mathrm{tr}\,[\mathbf{A}_{,1}\mathbf{A}_{,2}],$$
$$R_{ii} = \frac{1}{4}\mathrm{tr}\,[\mathbf{A}_{,i}\mathbf{A}_{,i}] - \frac{\alpha_i f_i}{\alpha f} + \frac{\alpha_{ii}}{\alpha} - \frac{\alpha_{2\,i}}{\alpha^2}$$
$$R_{13} = s\left[(\mathbf{A}_{,1})^2_2 - (\mathbf{A}_{,1})^1_1\right],\qquad R_{23} = -2s\,(\mathbf{A}_{,1})^1_2$$
$$R_{14} = s\left[(\mathbf{A}_{,2})^2_2 - (\mathbf{A}_{,2})^1_1\right],\qquad R_{24} = -2s\,(\mathbf{A}_{,2})^1_2.$$

If $\det\mathbf{F} > 0$, then the components of the Ricci tensor in a non-holonomic adapted basis are

$$R_{13} = s\left[(\mathbf{A}_{,1})^2_2 - (\mathbf{A}_{,1})^1_1\right];$$
$$R_{23} = s\left[(\mathbf{A}_{,2})^2_2 - (\mathbf{A}_{,2})^1_1\right]$$
$$R_{14} = -2s\,(\mathbf{A}_{,1})^1_2,\qquad R_{24} = -2s\,(\mathbf{A}_{,2})^1_2$$
$$(R_{ab}) = \frac{\mathbf{H}}{4f\alpha}\left[(\alpha\mathbf{A}_{,1})_{,1} + (\alpha\mathbf{A}_{,2})_{,2}\right] + \frac{s^2}{\alpha^2}\mathbf{H}h_{22}$$
$$R_{11} = \frac{1}{2}\Delta(\ln\alpha\ln|f|) + \frac{1}{4}\mathrm{tr}\,(\mathbf{A}_{,1})^2 + \frac{\alpha_{,2}f_{,2}}{2\alpha f} - \frac{\alpha_{,1}f_{,1}}{2\alpha f} + \left(\frac{\alpha_{,1}}{2\alpha}\right)_{,1} - \left(\frac{\alpha_{,2}}{2\alpha}\right)_{,2};$$
$$R_{22} = \frac{1}{2}\Delta(\ln\alpha\ln|f|) + \frac{1}{4}\mathrm{tr}\,(\mathbf{A}_{,2})^2 + \frac{\alpha_{,1}f_{,1}}{2\alpha f} - \frac{\alpha_{,2}f_{,2}}{2\alpha f} + \left(\frac{\alpha_{,1}}{2\alpha}\right)_{,1} - \left(\frac{\alpha_{,2}}{2\alpha}\right)_{,2}$$
$$R_{12} = \partial_1\partial_2(\ln\alpha) + \frac{1}{4}\mathrm{tr}\,[\mathbf{A}_{,1}\mathbf{A}_{,2}] - \frac{\alpha_{,1}f_{,2}}{2\alpha f} - \frac{\alpha_{,2}f_{,1}}{2\alpha f}$$

where $A_{,i} = \mathbf{H}^{-1}\mathbf{H}_{,i}$, $\alpha = \sqrt{|\det\mathbf{H}|}$, $\Delta = \partial_1^2 + \partial_2^2$ is the Laplace operator and $\alpha = \sqrt{|\det\mathbf{H}|}$.

4 Einstein metrics

4.1 Einstein metrics when $\mathbf{g(Y,Y)} \neq \mathbf{0}$

In the considered class of metrics, vacuum Einstein equations, $R_{\mu\nu} = 0$, can be completely solved [8]. If the Killing field Y is not of *light type, i.e.* $g(Y,Y) \neq 0$, then in the adapted coordinates (x,y,p,q) the general solution is

$$g = f(\mathrm{d}x^2 \pm \mathrm{d}y^2) + \beta^2[(s^2k^2q^2 - 2slq + m)\mathrm{d}p^2 + 2(l - skq)\mathrm{d}p\mathrm{d}q + k\mathrm{d}q^2] \tag{6}$$

where $f = -\frac{1}{2s^2k}\triangle_\pm\beta^2$, and $\beta(x,y)$ is a solution of the *tortoise equation*

$$\beta + A\ln|\beta - A| = u(x,y),$$

the function u being a solution either of Laplace or d'Alembert equation, $\triangle_\pm u = 0$, $\triangle_\pm = \partial^2_{xx} \pm \partial^2_{yy}$, such that $(\partial_x u)^2 \pm (\partial_y u)^2 \neq 0$. The constants k, l, m are constrained by $km - l^2 = \pm 1$, $k \neq 0$.

4.1.1 Normal form of metrics when $\mathrm{g(Y,Y)} \neq 0$

The metrics (6) are all locally diffeomorphic to

$$g = \varepsilon_1\left(\left[1 - \frac{A}{r}\right]\mathrm{d}\tau^2 \pm \left[1 - \frac{A}{r}\right]^{-1}\mathrm{d}r^2\right) + \varepsilon_2 r^2\left[\mathrm{d}\vartheta^2 + \mathcal{F}(\vartheta)\,\mathrm{d}\varphi^2\right] \tag{7}$$

where $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$ with a choice coherent with the required *signature* 2, (ϑ, φ) are *geographic coordinates* along Killing leaves, $\mathcal{F}(\vartheta)$ is equal either to $\sinh^2 \vartheta$ or $-\cosh^2 \vartheta$, depending on the signature of the metric, $r = 2s^2 k\beta$, and $\tau = v$ is the harmonic function conjugate to u.

The above form is described in the context of warped solutions in [4]. The geometric reason for this form is that, when $g(Y, Y) \neq 0$, a third Killing field exists which together with X and Y constitute a basis of $so(2, 1)$. The larger symmetry implies that the geodesic equations describe a *non-commutatively integrable system* [7], and the corresponding geodesic flow projects on the geodesic flow of the metric restricted to the Killing leaves. *The above local form does not allow, however, to treat properly the singularities appearing inevitably in global solutions.* Moreover, the metrics (6) play a relevant role in the construction of new global solutions as described in [8].

In analogy with what is done for the Schwarzschild solution, a distribution of matter giving rise to the static gravitational field

$$g = -f(r)\mathrm{d}\tau^2 + h(r)\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sinh^2\theta \mathrm{d}\varphi^2) \quad (8)$$

which reduces in the vacuum to the metric (7) may be also introduced [11].

4.2 Einstein metrics when $\mathbf{g(Y, Y) = 0}$

If the Killing field Y is of *light type*, then the general solution of vacuum Einstein equations, in the adapted coordinates (x, y, p, q), is given by

$$g = 2f(\mathrm{d}x^2 + \mathrm{d}y^2) + \mu[(w(x, y) - 2sq)\mathrm{d}p^2 + 2\mathrm{d}p\mathrm{d}q], \quad (9)$$

where $\mu = D\Phi + B$; $D, B \in \mathcal{R}$, Φ is a non constant harmonic function, $f = \pm (\nabla\Phi)^2 / \sqrt{|\mu|}$, and $w(x, y)$ is a solution of the *Darboux equation:*

$$\Delta w + (\partial_x \ln|\mu|)\, \partial_x w + (\partial_y \ln|\mu|)\, \partial_y w = 0.$$

The physical properties of the above gravitational wave will be described in a forthcoming paper [2].

The new solutions (9) together with (6) exhaust all local Lorentzian Ricci-flat metrics invariant for a $\mathcal{G}_2$ Lie algebra.

5 Global solutions

Any of previous metrics is fixed by a solution of the wave or Laplace equation, and a choice

– of the constant A and one of the branches of a solution of the tortoise equation, if $g(X, Y) \neq 0$;
– of a solution of Darboux equation, if $g(X, Y) = 0$.

The metric manifold (M, g) has a bundle structure whose fibers are the Killing leaves and whose base $\mathcal{W}$ is a bidimensional manifold diffeomorphic to the orthogonal leaves. The wave and Laplace equations mentioned above are defined on $\mathcal{W}$. Thus, the problem of the extension of our local solutions is reduced to that of the extension of $\mathcal{W}$. Such an extension carries a geometric structure[8], the $\mathbf{j}$-*complex structure*, that gives an intrinsic sense to the notion of the wave and Laplace equations and clarifies what variety of different geometries is, in fact, obtained.

Thus, any global metric is associated with a pair consisting of a $\mathbf{j}$-*complex curve* $\mathcal{W}$ and a $\mathbf{j}$-*harmonic function* u on it.

It will be now described in detail how to construct global solutions in the case in which $\mathcal{K}il\,(g_\Sigma)$ is $so\,(3)$ or $so\,(2, 1)$. The remaining cases can be found in [8].

Denote by (Σ, g_Σ) a homogeneous bidimensional Riemannian manifold, whose Gauss curvature $K\,(g_\Sigma)$, if different from zero, is normalized to ± 1. Let $(\mathcal{W}, u)$ be a pair consisting of a $\mathbf{j}$-complex curve $\mathcal{W}$ and a $\mathbf{j}$-harmonic function u on $\mathcal{W}$. The bundle structure $\pi_1 : M \to \mathcal{W}$ canonically splits in the product $\mathcal{W} \times \Sigma$. Denote by $\pi_2 : M \to \Sigma$ the also natural projection of $M = \mathcal{W} \times \Sigma$ on Σ. Then, the above data determine the following Ricci-flat manifold (M, g) with

$$g = \pi_1^* \left(g_{[u]}\right) + \pi_1^* \left(\beta^2\right) \pi_2^* \left(g_\Sigma\right) \quad (10)$$

where $\beta\,(u)$ is implicitly determined by the tortoise equation, and

$$g_{[u]} = \pm\beta^{-1}(\beta - A) \left(\mathrm{d}u^2 - \mathbf{j}^2 \mathrm{d}v^2\right).$$

6 Examples

Physically interesting examples and a procedure to construct new global solutions starting from local known ones, can be found in [8].

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